Nonreal eigenvalues for second order differential operators on networks with circuits *

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It is shown that on every finite network with at least one circuit there exist second order differential operators having an infinite number of nonreal eigenvalues. The presence of nonreal eigenvalues implies that these operators cannot be selfadjoint with respect to any metric. These eigenvalues reveal also the existence of oscillatory solutions for the corresponding time–dependent partial differential equations.

1. INTRODUCTION AND MAIN RESULT

The eigenvalues for Sturm–Liouville problems –that is, second order differential operators on the interval with *separated* boundary conditions– have been widely studied since its arising in the 1830's. The spectral properties for the same operators with non separated conditions, in particular the so–called periodic problem, are also well known (see for instance [6]). Many relevant properties of these problems are linked to the selfadjointness with respect to the metric induced by a suitable inner product.

This paper is part of a wider series of works devoted to studying problems of Sturm–Liouville type defined on a different kind of domains, the finite networks, which can be viewed as finite sets of thin beams or wires, having

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certain endpoints, called nodes, in common. These networks are usually identified as finite *graphs*, the wires are called *edges* and the nodes are called also *vertices*.

These problems appear, for instance, when studying the heat conduction, or more generally the diffusion and advection of a substance on this kind of objects. The conditions to be imposed at the nodes (see (2) and (3) below), also called transmission conditions, are quite natural from the physical point of view. Neumann or Dirichlet conditions are imposed at the *free* endpoints, whereas at the nodes the conditions are of Kirchhoff law type, which establish a balance among the fluxes at each node, similarly to the conditions at the nodes in an electrical network.

This subject can also be seen inside the wider context of partial differential equations on multistructures (see [1]), that have been a subject of increasing interest in the recent years, in relation with several problems arising in physics, engineering, chemistry and neurobiology. We may refer to the works of F. Ali Mehmeti, J. von Below, R. Carlson, G. Lumer, S. Nicaise and others. See especially [5].

Networks seem to be intermediate domains between dimension one and higher dimensions if we look at the properties of the spectra of linear differential operators defined on them. One of the features is that the geometric multiplicity of the eigenvalues, which is always 1 for the classical Sturm–Liouville problem, can be larger than 1 for networks (see [7]).

In the present work we show the different behaviour of the networks having at least one circuit from those without any circuit when we ask ourselves about the existence of nonreal eigenvalues for this kind of operators. The presence of nonreal eigenvalues reveals the existence of oscillatory solutions for the corresponding time–dependent parabolic equations, such as the diffusion and advection equations mentioned above.

So let us consider a connected and finite network $\mathcal{G}$, with M edges and N nodes. We refer to [11] for graph terminology. Nodes with degrees 1 and larger than 1 will be called *exterior* and *interior* nodes, respectively. By means of a convenient $\mathcal{C}^2$–parametrization (see [2]) we can identify each edge as a real interval. Let us call I_i $(i = 1, \ldots, M)$ the intervals identifying the edges of $\mathcal{G}$. We will understand that a function u defined on $\mathcal{G}$ is a M-vector $(u_1, \ldots, u_M)$, where each u_i is a function defined on I_i.

Let $L^2(\mathcal{G})$ be the space of functions $u = (u_1, \ldots, u_M)$ defined on $\mathcal{G}$, such that u_i is in $L^2(I_i)$ for all $i \in \{1, \ldots, M\}$, and consider the operator L of $L^2(\mathcal{G})$ such that Lu has the components

$$a^i(x)u_i''(x) + b^i(x)u_i'(x) + c^i(x)u_i(x) \quad (i = 1, ..., M), \tag{1}$$

(where a^i, b^i, c^i are suitable functions and $a^i(x) \geq \epsilon > 0$) with domain $H_b^2(\mathcal{G})$, the space of functions $u = (u_1, \ldots, u_M)$ defined on $\mathcal{G}$ such that each u_i is in $H^2(I_i)$ and verifies the *continuity conditions*

$$u_{j1}(e_{j1}) = u_{j2}(e_{j2}) = \ldots = u_{jk}(e_{jk}) \tag{2}$$

for every interior node j in which the edges $j1$, $j2$,...,jk are confluent by coincidence of their endpoints $e_{j1}, e_{j2}, \ldots, e_{jk}$, and also the *third class conditions* or *generalized Kirchhoff conditions*

$$\sum_{i=1}^{k} \alpha_{ji} u_{ji}^{(e)}(e_{ji}) + \beta_j u(j) = 0 \tag{3}$$

at every node, where $u^{(e)}$ means exterior derivative or towards the coincidence node j, and $u(j)$ is the common value at j, according to (2). In (3) we assume $\alpha_{ji} > 0$ for the interior nodes, and $\alpha_j \geq 0$, $\alpha_j^2 + \beta_j^2 > 0$ for the exterior nodes. For interior nodes the condition is simply called *Kirchhoff condition* if $\alpha_{ji} = 1$ and $\beta_j = 0$. For an exterior node j we will have Neumann condition if $\beta_j = 0$, and Dirichlet condition if $\alpha_j = 0$. Without loss of generality we can suppose $I_i = [0, 1]$, $(i = 1, ..., M)$.

This operator L allows us to write in the abstract form $du/dt = L\,u$ the parabolic system of partial differential equations

$$\frac{\partial u_i}{\partial t} = a^i(x)\frac{\partial^2 u_i}{\partial x^2} + b^i(x)\frac{\partial u_i}{\partial x} + c^i(x)u_i(x) \quad (i = 1, ..., M)$$

with the coupled boundary conditions (2) and (3). In particular, we see that the existence of nonreal eigenvalues for $L\,u_0 = \lambda u_0$ implies the oscillatory behavior of the solutions $u(t, x) = e^{\lambda t}u_0(x)$.

The expression (1) can be written in the *formally selfadjoint* version

$$\frac{1}{r_i(x)}(p_i(x)u_i'(x))' + q_i(x)u_i(x) \quad (i = 1, ..., M),$$

where

$$p_i(x) = k_i \exp\left(\int_0^x \frac{b^i(t)}{a^i(t)}\,dt\right), \quad r_i(x) = \frac{p_i(x)}{a^i(x)}, \tag{4}$$

with k_i an arbitrary positive constant for each edge.

We define in the Hilbert space $L^2(\mathcal{G})$ the scalar product

$$(u, v)_r = \int_{\mathcal{G}} r(x)u(x)\overline{v}(x) = \sum_{i=1}^{M}\int_0^1 r_i(x)u_i(x)\overline{v_i}(x)\,dx \tag{5}$$

which induces a norm that is equivalent to the usual one of $L^2(\mathcal{G})$ (notice that this scalar product depends on the choice of the k_i in (4)).

We recall the following results from [9] and [10]. The first of them gives us a criterion for the symmetry of L with respect to the product (5). The second one establishes the selfadjointness, with respect to (5), of every operator of this kind for the special case in which $\mathcal{G}$ is a tree, and the third one gives an additional condition to be fulfilled by graphs with circuits in order to obtain selfadjointness with respect to (5).

I. THEOREM. *For a given choice of the k_i in* (4)*, L is symmetric with respect to* (5) *if and only if for every interior node j there exists μ_j such that $p_i(e_{ji}) = \mu_j \alpha_{ji}$ for all the endpoints e_{ji} of edges incident to j.*

II. THEOREM. *If $\mathcal{G}$ is a tree, then there exists a choice for the k_i in* (4) *such that L is selfadjoint for the metric induced by* (5)*.*

Let us suppose now that $\mathcal{G}$ contains at least one circuit. Let $\mathcal{C}$ be one of these circuits, having $M_{\mathcal{C}}$ edges and, obviously, $M_{\mathcal{C}}$ nodes. Let us order them by putting that the edge i connects the nodes i and $i+1$. We denote as $\alpha_{ij}^{\mathcal{C}}$ the coefficient of the generalized Kirchhoff condition for the edge i at the node j. By running along the circuit we find the coefficients $\alpha_{11}^{\mathcal{C}}$, $\alpha_{12}^{\mathcal{C}}$, $\alpha_{22}^{\mathcal{C}}$, $\alpha_{23}^{\mathcal{C}}$, ..., $\alpha_{M_{\mathcal{C}},1}^{\mathcal{C}}$.

III. THEOREM. *If $\mathcal{G}$ is not a tree, then the necessary and sufficient condition for the existence of a choice of the k_i in* (4) *such that L is selfadjoint for the metric induced by* (5) *is the following:*

$$\sum_{i=1}^{M_{\mathcal{C}}} \int_0^1 \frac{b^i(x)}{a^i(x)}\, dx = \sum_{i=1}^{M_{\mathcal{C}}} \ln \frac{\alpha_{i,i+1}^{\mathcal{C}}}{\alpha_{ii}^{\mathcal{C}}}$$

for every circuit $\mathcal{C}$ of $\mathcal{G}$, where i has to be taken modulo $M_{\mathcal{C}}$.

The main result of this paper is the following theorem, which shows the qualitative different behaviour of trees and graphs with circuits with respect to the selfadjointness of these operators.

THEOREM 1.1. *For every connected and finite network having at least one circuit we can find operators of the kind defined above that have an infinite number of nonreal eigenvalues.*

We have to point out that in the theorems I–III we worked with a metric that is induced in a quite natural way by the coefficients of the differential equations on each edge. The symmetry condition obtained in I above, also called *consistency* condition, depends heavily on the metric (5). There is the possibility of working with other metrics, and we refer to [4], where it is shown that for certain networks is possible to have real spectra even in the case that the consistency condition is not fulfilled. J. von Below in [2] was the first to exhibit an example, in a very simple graph, of a *formally selfadjoint* differential operator having nonreal eigenvalues due to inconsistent Kirchhoff conditions. This feature was later exploited more extensively by the authors in [8], where the graph is a single circuit. In the present paper our result shows that this is also possible for arbitrary graphs containing at least one circuit.

The main result of this paper shows also that the only networks where every operator of this kind is selfadjoint in some metric are the trees, as considered in II above.

2. PROOF OF THEOREM 1.1

Let $\mathcal{G}_0$ be a connected simple network with N nodes, that is, without loops and without multiple edges. A widely used tool to analyze the geometry of $\mathcal{G}_0$ is the so-called Laplacian matrix $\mathcal{L} = (\alpha_{i,j})_{i,j=1,\ldots,N}$ of $\mathcal{G}_0$, defined by

$$\alpha_{i,i} = \text{ degree of node } i$$
$$\alpha_{i,j} = -1, \text{ if } i \neq j \text{ and there is an edge connecting nodes } i \text{ and } j$$
$$\alpha_{i,j} = 0, \text{ otherwise.}$$

As it is easy to see, $\det \mathcal{L} = 0$, but every collection of $N-1$ columns of $\mathcal{L}$ are linearly independent.

By using this matrix $\mathcal{L}$ we make the following auxiliary statement, that will be essential in the proof of Theorem 1.1.

Lemma 2.1. *Let $\mathcal{G}_0$ and $\mathcal{L}$ be as above. Let us take an edge of $\mathcal{G}_0$ and suppose that it connects the nodes of indices k and l. Let $0 < \epsilon < \delta$ be two real numbers with the property that $\delta - \epsilon = \delta\epsilon$, and let us define the matrix $(\beta_{i,j})_{i,j=1,\ldots,N}$ by: $\beta_{k,l} = -\delta$, $\beta_{l,k} = \epsilon$, and $\beta_{i,j} = 0$ otherwise. Then we claim that* $\det(\mathcal{L} + (\beta_{i,j})) < 0$ *if the edge connecting the nodes k and l is not a bridge (i.e. if it belongs to a circuit of $\mathcal{G}_0$), and* $\det(\mathcal{L} + (\beta_{i,j})) = 0$ *otherwise.*

Proof. Without loss of generality, we can suppose that $k = N-1$ and $l = N$. We consider the little more general determinant

$$D = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N-1} & a_{1,N} \\ a_{1,2} & a_{2,2} & \cdots & a_{2,N-1} & a_{2,N} \\ \dots & \dots & \ddots & \dots & \dots \\ a_{1,N-1} & a_{2,N-1} & \cdots & a_{N-1,N-1} & a_{N-1,N}-\delta \\ a_{1,N} & a_{2,N} & \cdots & a_{N-1,N}+\epsilon & a_{N,N} \end{vmatrix}$$

for a symmetric $N \times N$ matrix $(a_{i,j})$ with the following properties:

i) Every diagonal element $a_{i,i}$ is strictly positive.
ii) Every non-diagonal element $a_{i,j}$, $i \neq j$, is negative or zero.
iii) The sum of all the elements of every row is zero:

$$\sum_{j=1}^{N} a_{i,j} = 0, \text{ for all } i = 1, 2, \dots, N.$$

iv) Every collection of $N-1$ columns of $(a_{i,j})$ are linearly independent.

We observe that the matrix $(a_{i,j}) = (\alpha_{i,j}) = \mathcal{L}$ satisfies all of these properties, and also verifies that

v) A non-diagonal element $a_{i,j}$ is strictly less than zero (and equal to -1) if and only if there is an edge connecting the nodes i and j.

We are going to calculate this determinant by a Gauss triangularization. Let us sum a multiple of the first column to the other ones, in such a way that the first element in such columns becomes zero, and develop the determinant by the first row. The result is that D is the product of $a_{1,1}$ (> 0) by the determinant

$$D' = \begin{vmatrix} a'_{2,2} & a'_{2,3} & \cdots & a'_{2,N-1} & a'_{2,N} \\ a'_{2,3} & a'_{3,3} & \cdots & a'_{3,N-1} & a'_{3,N} \\ \dots & \dots & \ddots & \dots & \dots \\ a'_{2,N-1} & a'_{3,N-1} & \cdots & a'_{N-1,N-1} & a'_{N-1,N}-\delta \\ a'_{2,N} & a'_{3,N} & \cdots & a'_{N-1,N}+\epsilon & a'_{N,N} \end{vmatrix}$$

for the symmetric $(N-1) \times (N-1)$ matrix $(a'_{i,j})_{i,j=2,\dots,N}$ defined by

$$a'_{i,j} = a'_{j,i} = a_{i,j} - \frac{a_{1,i}a_{1,j}}{a_{1,1}} \leq a_{i,j} \quad (i, j = 2, \dots, N).$$

Observe that $a'_{i,j} < a_{i,j}$ if and only if both $a_{1,i}$ and $a_{1,j}$ are nonzero.

We have also

$$\sum_{j=2}^{N} a'_{i,j} = \sum_{j=2}^{N} a_{i,j} - \frac{a_{1,i}}{a_{1,1}} \sum_{j=2}^{N} a_{1,j} = -a_{i,1} - \frac{a_{1,i}}{a_{1,1}}(-a_{1,1}) = 0 \quad (i = 2, \dots, N).$$

So $(a'_{i,j})$ also satisfies ii) and iii). It is also clear that it satisfies iv) for every collection of $N-2$ columns, and, in particular, this implies that there are no columns identically zero. From this, the symmetry, and the properties ii) and iii), the property i) follows. So D' is a determinant of the same type as D. But from the formula

$$a'_{i,j} = a'_{j,i} = a_{i,j} - \frac{a_{1,i}a_{1,j}}{a_{1,1}},$$

when $(a_{i,j}) = (\alpha_{i,j}) = \mathcal{L}$ we see that instead of v) it satisfies that

v$'$) A non-diagonal element $a'_{i,j}$ is strictly less than $\alpha_{i,j}$ if and only if there is a 2-path in $\mathcal{G}_0$ connecting the nodes i and j through the node 1.

Iterating this, we arrive, after k steps, to the fact that D is the product of the positive numbers $a_{1,1}$, $a'_{2,2}$, $a''_{3,3}$, …, $a^{(k-1)}_{k,k}$ by $D^k = \det(a^{(k)}_{i,j})$, where $(a^{(k)}_{i,j})_{i,j=k+1,\dots N}$ also satisfies i), ii), iii), and also iv) for every collection of $N-k-1$ columns. For the case $(a_{i,j}) = (\alpha_{i,j}) = \mathcal{L}$, one inductively sees that it also satisfies the following property:

v^k) A non-diagonal element $a^k_{i,j}$ is strictly less than $\alpha_{i,j}$ if and only if there is a path in $\mathcal{G}_0$ of at least 2 edges connecting the nodes i and j only along some of the nodes $1, 2, \dots, k$.

To clarify this fact, observe, for example, that when $k = 2$ the elements that satisfy $a''_{i,j} < \alpha_{i,j}$ are exactly those corresponding to nodes i and j such that either both nodes are directly connected to node 1, or both are directly connected to node 2, or, when node 2 is directly connected to node 1, if one node is directly connected to node 1 and the other to node 2 (or if more than one of these possibilities happens at the same time).

After $N-2$ steps, we see that D is the product of the positive numbers $a_{1,1}$, $a'_{2,2}$, $a''_{3,3}$, …, $a^{(N-3)}_{N-2,N-2}$ by the 2-determinant (with the same properties)

$$\begin{vmatrix} a^{(N-2)}_{N-1,N-1} & a^{(N-2)}_{N-1,N} - \delta \\ a^{(N-2)}_{N-1,N} + \epsilon & a^{(N-2)}_{N,N} \end{vmatrix},$$

whose value is

$$a^{(N-2)}_{N-1,N-1} a^{(N-2)}_{N,N} - (a^{(N-2)}_{N-1,N} - \delta)(a^{(N-2)}_{N-1,N} + \epsilon)$$

$$
\begin{aligned}
&= a^{(N-2)}_{N-1,N} a^{(N-2)}_{N-1,N} - a^{(N-2)}_{N-1,N} a^{(N-2)}_{N-1,N} + a^{(N-2)}_{N-1,N}(\delta - \epsilon) + \delta\epsilon \\
&= (a^{(N-2)}_{N-1,N} + 1)(\delta\epsilon)
\end{aligned}
$$

(by using property iii) and the fact that $\delta - \epsilon = \epsilon\delta$). This is clearly less than or equal to zero, and it is negative if and only if $a^{(N-2)}_{N-1,N} < -1$. But, because of the property v^{N-2}) and the fact that $\alpha_{N-1,N} = -1$ we see that this happens if and only if the node $N-1$ can be connected to the node N by a path of at least 2 edges only through some of the nodes $1, 2, \ldots, N-2$. This is to say, if and only if the edge connecting $N-1$ and N belongs to a circuit. And this finishes the proof of the lemma. ▌

2.1. Proof of Theorem 1.1

Proof. Let $\mathcal{G}$ be a network having at least one circuit. If $\mathcal{G}$ has any loop, we transform it into a circuit by creating two nodes inside the loop. If $\mathcal{G}$ has multiple edges, we put new nodes inside them. The resulting network $\mathcal{G}_0$ has no loops and the connections between nodes are unique.

Let us suppose that $\mathcal{G}_0$ has M edges and N nodes, and consider an edge e_k belonging to a circuit. We put in every edge of $\mathcal{G}_0$ one new node and orientate the resulting $2M$ edges in such a way that, identified all of them as copies of the interval $[0,1]$, every original edge transforms into a couple of intervals $[0,1]$ with opposite orientations and joined by their 1-endpoints by means of the new nodes, which will be called 1-nodes in the following. In the old nodes, exterior ones included, we have only 0-endpoints, and they will be called 0-nodes in the following. These nodes have the same degree that they had in $\mathcal{G}$. All the 1-nodes have degree 2. The resulting network has $2M$ edges and $N+M$ nodes.

On each of these intervals $[0,1]$, we consider the equation

$$u_j'' + \lambda u_j = 0, \tag{6}$$

except for the couple corresponding to e_k, in which we consider

$$u'' + 2ku' + (\lambda + k^2)u = 0 \quad \text{and} \quad u'' - 2ku' + (\lambda + k^2)u = 0, \tag{7}$$

where $k > 0$, and we impose continuity and Kirchhoff conditions at the interior nodes and Neumann conditions at the exterior ones. We observe that in the original network $\mathcal{G}_0$ (6) and (7) correspond to

$$u_j'' + \lambda u_j = 0 \quad \text{and} \quad u'' + 2ku' + (\lambda + k^2)u = 0, \tag{8}$$

respectively.

Let us write the general solution of the equations (6) and (7) in the form $u_j = A_j\phi_j + B_j\psi_j$, where ϕ_j and ψ_j are the solutions such that $\phi_j(0) = 1,\ \phi_j'(0) = 0,\ \psi_j(0) = 0,\ \psi_j'(0) = 1$. The functions ϕ_j and ψ_j are easily computed in terms of hyperbolic functions.

The continuity and Kirchhoff conditions at an interior 0-node ν receiving the edges $\nu_1, \nu_2, \ldots, \nu_{m_\nu}$ are written as

$$A_{\nu_1} = A_{\nu_2} = \ldots = A_{\nu_{m_\nu}} =: A_\nu, \quad B_{\nu_1} + B_{\nu_2} + \ldots + B_{\nu_{m_\nu}} = 0;$$

and for an exterior node ν_e the Neumann condition reads $B_{\nu_e} = 0$.

In the following we will use the notation A_i for the common A for all the edges incident to the 0-node i, and B_{ij}, B_{ji} for the B's of the couple of edges starting, respectively, at the 0-nodes i and j and ending at the same 1-node. Obviously, B_{ij} and B_{ji} do not exist if there is no connection between i and j in the original network.

Then the continuity and Kirchhoff conditions for a 1-node are written as

$$A_i\phi_{ij}(1) + B_{ij}\psi_{ij}(1) = A_j\phi_{ji}(1) + B_{ji}\psi_{ji}(1)$$

$$A_i\phi_{ij}'(1) + B_{ij}\psi_{ij}'(1) + A_j\phi_{ji}'(1) + B_{ji}\psi_{ji}'(1) = 0.$$

We will suppose that we have ordered the 0-nodes and edges in such a way that the couple of e_k is the last one.

By finding the ϕ's and ψ's for (6) and (7) and writing the boundary conditions we obtain an algebraic linear system having $N + 2M$ equations and $N + 2M$ unknowns. The determinant of this system is a function of the complex variable λ, and the zeroes of this function are the eigenvalues of the problem. Let us order the rows and columns of the matrix $\mathcal{M}(\lambda)$ of this system in the following way:

- For the columns the order will be A_1, …, A_N, B_{11}, …, B_{1N}, (remember that in this list B_{ij} and B_{ji} do not exist if there is no connection between i and j or if $i = j$).

- For the rows, we write first the corresponding to Kirchhoff or Neumann conditions at the 0-nodes, and after that the rows of the continuity and Kirchhoff conditions at the 1 nodes, putting at the end those that correspond to the connection between the two edges of e_k.

The first N rows of $\mathcal{M}(\lambda)$ look as

A_1	A_2	$\dots$	A_N	B_{11}	$\dots$	B_{1N}	B_{21}	$\dots$	B_{2N}	$\dots$	B_{N1}	$\dots$	B_{NN}
0	0	$\dots$	0	1	$\dots$	1	0	$\dots$	0	$\dots$	0	$\dots$	0
0	0	$\dots$	0	0	$\dots$	0	1	$\dots$	1	$\dots$	0	$\dots$	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
0	0	$\dots$	0	0	0	$\dots$	0	$\dots$	0	$\dots$	1	$\dots$	1

The rest of rows of $\mathcal{M}(\lambda)$ are in couples. Every couple has eight nonzero (in general) elements that appear (except for the last couple) in the matrix in the following way, in which we abbreviate C for $\cosh m$ and S for $\sinh m$, and we have put for simplicity $m = \sqrt{-\lambda}$:

$\dots$	A_i	$\dots$	A_j	$\dots$	B_{ij}	$\dots$	B_{ji}	$\dots$
$\cdots$	C	$\cdots$	$-C$	$\cdots$	S/m	$\cdots$	$-S/m$	$\cdots$
$\cdots$	mS	$\cdots$	mS	$\cdots$	C	$\cdots$	C	$\cdots$

The two last rows read

$\dots$	A_{N-1}	A_N	$\dots$	$B_{N-1,N}$	$\dots$	$B_{N,N-1}$
$\cdots$	$e^{-k}(C + \frac{kS}{m})$	$-e^{k}(C - \frac{kS}{m})$	$\cdots$	$e^{-k}\frac{S}{m}$	$\cdots$	$-e^{k}\frac{S}{m}$
$\cdots$	$e^{-k}(m - \frac{k^2}{m})S$	$e^{k}(m - \frac{k^2}{m})S$	$\cdots$	$e^{-k}(C - \frac{kS}{m})$	$\cdots$	$e^{k}(C + \frac{kS}{m})$

We will show that $\det \mathcal{M}(\lambda)$ has an infinite number of nonreal zeroes. We exclude the cases $\sinh m = 0$ and $\cosh m = 0$, because they give real values of λ.

Let us write (A_i), (B_{ij}) for the columns corresponding to A_i, B_{ij}. We substitute each column (A_i) by

$$(A_i) - \frac{m}{2CS}\left(\sum_{j=1}^{N}(C^2 + S^2 + 2k_{ij}SC/m)(B_{ij}) - \sum_{j=1}^{N} e^{2k_{ji}}(B_{ji})\right)$$

where $k_{ij} = 0$ except for $k_{N,N-1} = k$ and $k_{N-1,N} = -k$.

The matrix $\mathcal{M}$ becomes (without changing the value of the determinant)

$$\begin{pmatrix} (\mathcal{M}_1) & (\mathcal{M}_3) \\ (\,0\,) & (\mathcal{M}_2) \end{pmatrix},$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are square matrices, thus $\det(\mathcal{M}) = \det(\mathcal{M}_1)\det(\mathcal{M}_2)$.

$\mathcal{M}_1$ is a $N \times N$ matrix whose elements d_{ij} are, using that $C^2 + S^2 = \cosh(2m)$ and $2CS = \sinh(2m)$, and calling n_i the number of edges incident to the node i,

$$\begin{aligned}
d_{ii} &= -\frac{m}{\sinh(2m)} n_i \cosh(2m) \quad (i = 1, \dots, N-2),\\
d_{N-1,N-1} &= -\frac{m}{\sinh(2m)} (n_{N-1}\cosh 2m - \frac{k}{m}\sinh 2m),\\
d_{NN} &= -\frac{m}{\sinh(2m)} (n_N \cosh 2m + \frac{k}{m}\sinh 2m),\\
d_{ij} &= 0 \quad \text{if } i \neq j \text{ and there is no connection } i-j,\\
d_{ij} &= \frac{m}{\sinh(2m)} \quad \text{if } i \neq j \text{ and there is connection } i-j, \text{ except for}\\
d_{N-1,N} &= \frac{me^k}{\sinh(2m)} \quad \text{and}\\
d_{N,N-1} &= \frac{me^{-k}}{\sinh(2m)}.
\end{aligned}$$

Let us put $x = \cosh 2m$, $y = \sinh 2m$, suppose $k > 0$ and put also $\delta = e^k - 1 > 0$ and $\epsilon = 1 - e^{-k} > 0$. We have

$$\det(\mathcal{M}_1) = \left(-\frac{m}{y}\right)^N \begin{vmatrix} n_1 x & a_{12} & \cdots & a_{1,N-1} & a_{1N} \\ a_{12} & n_2 x & \cdots & a_{2,N-1} & a_{2N} \\ \dots & \dots & \ddots & \dots & \dots \\ a_{1,N-1} & a_{2,N-1} & \cdots & n_{N-1}x - \frac{k}{m}y & -1-\delta \\ a_{1N} & a_{2N} & \cdots & -1+\epsilon & n_N x + \frac{k}{m}y \end{vmatrix},$$

where $a_{ij} = a_{ji} = -1$ or 0 for connection or not between i and j, except for $a_{N-1,N}$ and $a_{N,N-1}$.

In $\mathcal{M}_2$ we reorder the columns in such a way that every couple (B_{ij}), (B_{ji}) appear together (in the last place, the couple $(B_{N-1,N})$, $(B_{N,N-1})$). Then the matrix $\mathcal{M}_2$ looks

$$\begin{pmatrix} (\mathcal{M}'_1) & (\ 0\ ) & \cdots & (\ 0\ ) \\ (\ 0\ ) & (\mathcal{M}'_2) & \cdots & (\ 0\ ) \\ \vdots & \vdots & \ddots & \vdots \\ (\ 0\ ) & (\ 0\ ) & \cdots & (\mathcal{M}'_M) \end{pmatrix},$$

where

$$\mathcal{M}'_i = \begin{pmatrix} S/m & -S/m \\ C & C \end{pmatrix}$$

for $i = 1, \ldots, M-1$, and

$$\mathcal{M}'_M = \begin{pmatrix} e^{-k}S/m & -e^{k}S/m \\ e^{-k}(C - kS/m) & e^{k}(C + kS/m) \end{pmatrix};$$

therefore, up to possible changes of sign,

$$\det(\mathcal{M}_2) = \det(\mathcal{M}'_1)\det(\mathcal{M}'_2)\cdots\det(\mathcal{M}'_M) = (2SC/m)^M = (y/m)^M,$$

and then $\det(\mathcal{M}) = (\frac{y}{m})^{M-N}\mathcal{D}(\lambda)$, where

$$\mathcal{D}(\lambda) = \begin{vmatrix} n_1x & a_{12} & \cdots & a_{1,N-1} & a_{1N} \\ a_{12} & n_2x & \cdots & a_{2,N-1} & a_{2N} \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ a_{1,N-1} & a_{2,N-1} & \cdots & n_{N-1}x - \frac{k}{m}y & -1-\delta \\ a_{1N} & a_{2N} & \cdots & -1+\epsilon & n_Nx + \frac{k}{m}y \end{vmatrix}.$$

So the values of λ for which $\mathcal{D}(\lambda)$ is zero are eigenvalues. Observe that $\mathcal{D}(\lambda) = \mathcal{D}(-m^2) = \mathcal{D}_0(m) + \mathcal{D}_1(m)/m + \mathcal{D}_2(m)/m^2$, where

$$\mathcal{D}_0(m) = \begin{vmatrix} n_1x & \cdots & a_{1,N-1} & a_{1N} \\ \ldots & \ddots & \ldots & \ldots \\ a_{1,N-1} & \cdots & n_{N-1}x & -1-\delta \\ a_{1N} & \cdots & -1+\epsilon & n_Nx \end{vmatrix},$$

$$\mathcal{D}_1(m) = \begin{vmatrix} n_1x & \cdots & 0 & a_{1N} \\ \ldots & \ddots & \ldots & \ldots \\ a_{1,N-1} & \cdots & -ky & -1-\delta \\ a_{1N} & \cdots & 0 & n_Nx \end{vmatrix} + \begin{vmatrix} n_1x & \cdots & a_{1,N-1} & 0 \\ \ldots & \ddots & \ldots & \ldots \\ a_{1,N-1} & \cdots & n_{N-1}x & 0 \\ a_{1N} & \cdots & -1+\epsilon & ky \end{vmatrix},$$

and

$$\mathcal{D}_2(m) = \begin{vmatrix} n_1x & \cdots & a_{1,N-1} & a_{1N} \\ \ldots & \ddots & \ldots & \ldots \\ a_{1,N-1} & \cdots & -ky & -1-\delta \\ a_{1N} & \cdots & -1+\epsilon & ky \end{vmatrix}.$$

Since $x = \cosh 2m$ and $y = \sinh 2m$, it turns out that the functions $\mathcal{D}_0(m), \mathcal{D}_1(m)$ and $\mathcal{D}_2(m)$ are entire functions of the complex variable m, and all of them are periodic of period $2\pi i$.

Let us consider the function $\mathcal{D}_0(m)$ when m is real. The biggest term of $\mathcal{D}_0(m)$ as $m \to \infty$ is $n_1n_2\cdots n_{N-1}n_Nx^N$, and so $\mathcal{D}_0(x)$ is positive for large values of m. By applying Lemma 1 we obtain that $\mathcal{D}(0)$ is strictly negative.

Therefore, $\mathcal{D}_0(m_0) = 0$ for some $m_0 > 0$. By the periodicity of $\mathcal{D}_0$, also $\mathcal{D}_0(m_0 + 2\pi in) = 0$ for all integer n. Since $\mathcal{D}_0(m)$ is not identically zero, there exists an $r_0 > 0$ such that if $0 < r < r_0$ then $|\mathcal{D}_0(m_0 + re^{i\theta})| > 0$ for all θ. So, if we call

$$d(r) = \inf\{|\mathcal{D}_0(m_0 + re^{i\theta})|; 0 \leq \theta < 2\pi\}$$

then $d(r) > 0$. By the periodicity of $\mathcal{D}_1$ and $\mathcal{D}_2$, we see that for $|n|$ large enough

$$\left| \frac{\mathcal{D}_1(m_0 + 2\pi in + re^{i\theta})}{m_0 + 2\pi in + re^{i\theta}} + \frac{\mathcal{D}_2(m_0 + 2\pi in + re^{i\theta})}{(m_0 + 2\pi in + re^{i\theta})^2} \right| < d(r)$$

for all θ.

Therefore, by Rouché's Theorem, the equation $\mathcal{D}(-m^2) = 0$ has at least one solution m_n in each of the infinitely many discs of center $m_0 + 2\pi in$ and radius r, with $|n|$ large enough. If this r is chosen small enough, these discs do not intersect the imaginary axis, so the eigenvalues $\lambda_n = -m_n^2$ are nonreal. They tend to infinity lying in between of the two parabolas

$$\text{Re } z = \left(\frac{\text{Im} z}{2(m_0 \pm r)} \right)^2 - (m_0 \pm r)^2.$$

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